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من قبل

وسن عباس جاسم الحسناوي

بإشراف

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١٤٢٧هـ ٢٠٠٦ م

REPUBLIC OF IRAQ
MINISTRY OF HIGHER EDUCATION
AND SCIENTIFIC RESEARCH
BABYLON UNIVERSITY
COLLEGE OF EDUCATION

Application of Central Limit Theorem on Probability Distributions by Using Differential Geometry

A THESIS
SUBMITTED TO THE DEPARTMENT OF MATHEMATICS- COLLEGE OF EDUCATION- UNIVERSITY OF BABYLON IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF MASTER OF SCIENCE IN MATHEMATICS

By
Wasan Abbas Jasim Al-Hisnawi

Supervised by
Assist. Prof. Hussein Ahmed Ali

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بسم الله الرحمن الرحيم

﴿ وَأَنْ لَيسَ لِلإِنسَانِ إِلاَّ مَا سَعَى* وَأَنَّ سَعْيَهُ سَوْفَ يُرَى* ثُمَّ يُجْزاهُ الجَزَاءَ الأَوفَى* وَأَنَّ إلى رَبِّكَ المنْتَهى ﴾

صدق الله العلي العظيم

سورة النجم آية (٣٩-٤٢)

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We certify that we have read this thesis entitled "**Application of Central Limit Theorem on Probability Distributions by Using Differential Geometry**" and, as an Examining Committee, we examined the student in its content, and what is related to it, and that in our opinion it is adequate with standing as a thesis for the degree of Master of Science in Mathematics.

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SUPERVISOR CERTIFICATION

I certify that this thesis was prepared under my supervision at the Department of Mathematics, College of Education, University of Babylon, as a partial fulfillment of the requirements for the degree of Master of Science in Mathematics.

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Date: / / 2006

RECOMMENDATION OF THE HEAD OF THE DEPARTMENT

In view of the available recommendations, I forward this research for debate by the Examining Committee.

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الاهـــــداء

الى ينبوع المحبة والتضحية والعطاء الدائم

والدي

الى نبع الحنان وشلال العطاء

الى القلب الذي يمدني بالدفء

امي الحبيبة

الى احباء قلبي..

إخواني وأخواتي

وسن

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الخلاصة

إن استخدام الرياضيات لوضع الحل لمسائل عديدة، معروف من الماضي. وفقاً لهذه الحقيقة تم دراسة استخدام الهندسة التفاضلية لتبيان اي من التوزيعات المستمرة يقترب من التوزيع الطبيعي وذلك باستخدام الربط بين الهندسة التفاضلية والاحصاء. وبالاخص توضيح الربط بين توزيع باريتو وتوزيع والد مع التوزيع الطبيعي باستخدام بعض النظريات الاحصائية والهندسة التفاضلية. وتتضمن الرسالة خمسة فصول:

الفصل الأول يقدم بعض التعاريف والمفاهيم من الهندسة التفاضلية مثل Tensor، الصيغة الأساسية الأولى والثانية، تقوس كاوس (Gaussian Curvature)، من هذه المفاهيم تم الحصول على صيغ مختلفة لحساب قيمة تقوس كاوس (Gaussian Curvature) على اساس العلاقة بين Riemannian Metric و Fisher Information.

الفصل الثاني يوضح بعض التعاريف والمفاهيم من الاحتمالية والاحصاء التي تحتاج اليها الرسالة مثل دالة الكثافة الاحتمالية، دالة التوزيع المستمرة، بعض التوزيعات المستمرة الخاصة، Fisher Information، تقارب المتغير العشوائي، قانون الاعداد الكبيرة ومبرهنة الغاية المركزية.

الفصل الثالث يقدم بعض العلاقات بين الإحصاء والهندسة التفاضلية، مثل تعريف المعاملات لـ Expected Fisher Information Matrix على أنها تساوي معاملات الصيغة الأساسية الأولى أو (Riemannian Metric) معطاة من قبل:

$$g_{ij} = -\int \frac{\partial^2 \log f(x,\theta)}{\partial\theta_i \ \partial\theta_j} f(x,\theta)dx = -E\left[\frac{\partial^2 \log f(x,\theta)}{\partial\theta_i \ \partial\theta_j}\right],$$

العلاقة بين Riemannian Metrics و Geodesics، Fisher Information Rao Distances بين التوزيعات الاحتمالية، Riemannian Metrics لبعض التوزيعات. تقوس كاوس (Gaussian Curvature) للتوزيعات الاحتمالية، وتم تعريف Christoffel symbols مع بعض الأمثلة لحساب تقوس كاوس باستخدام صيغ مختلفة.

في الفصل الرابع تم استخدام بعض الطرق لحساب Gaussian Curvature لبعض التوزيعات. كذلك تم تطبيق هذه الطرق لحساب Gaussian Curvature لتوزيع باريتو وتوزيع والد.

وتضمن الفصل الخامس الاستنتاجات والتوصيات.

Abstract

The using of mathematics to get a solution for many problems, is known from the past. According to truth we study the using of differential geometry to show which continuous distribution converges to normal distribution by connecting between differential geometry and statistics.

In particular, we illustrate the connection between Pareto distribution and Wald distribution with normal distribution by using some statistical theorems and differential geometry. The thesis consists of five chapters:

The first chapter introduces some definitions and concepts from differential geometry like tensor, the first and second fundamental form, Gaussian curvature, etc. From these concepts we get different formulas to calculate the value of Gaussian curvature based on the relation between Riemannian metrics and Fisher information.

The second chapter explains some definitions and concepts from probability and statistics needed in later chapters such as probability density function, continuous distribution function, some special continuous distributions, Fisher information, convergence of random variable, the law of large numbers and the central limit theorem.

The third chapter presents some connections between statistics and differential geometry, such as the definition of the coefficients of the expected Fisher information matrix as they equal to the coefficients of the first fundamental form (Riemannian metrics) given by:

$$g_{ij} = -\int \frac{\partial^2 \log f(x,\theta)}{\partial\theta_i \partial\theta_j} f(x,\theta)dx = -E\left[\frac{\partial^2 \log f(x,\theta)}{\partial\theta_i \partial\theta_j}\right],$$

the relation between the Riemannian metrics and geodesics, Fisher information Rao distances between probability distributions, Riemannian metrics for some distributions, the Gaussian curvature of the probability

distributions, and the Christoffel symbols. Some examples are also given to compute the Gaussian curvature using different formulas.

In chapter four we use some methods to calculate the Gaussian curvature for some distributions. Also, we apply these methods to calculate the Gaussian curvature for Pareto distribution and Wald distribution.

Chapter five contains some conclusions and recommendations.

CONTENTS

Introduction

The studies of applying the differential geometry in statistical subject are very little and do not cover each the sides, also the references under this title are limited. The knowledge of student about the differential geometry is very important for the M. Sc. study.

This thesis is divided into five chapters:

Chapter one contains some important concepts of differential geometry, like tensor, Riemannian metric, second fundamental form, Gaussian curvature, geodesics and curvature tensor.

Chapter two presents some important concepts and theorems of mathematical statistics, such as continuous random variable, some continuous distributions and Fisher information. Also, we show the theorems of law of the large numbers and central limit theorem.

Chapter three gives some interesting connections between statistics and differential geometry.

Chapter four contains the results of computing the Gaussian curvature (K) of some continuous distributions, like normal, Cauchy, t, gamma, Pareto and wald.

Chapter five contains some conclusions and the recommendations.

There are some researchers who worked in this field in end of twenty century and beginning of twenty one century:

In 1986, Barndorff- Nielsen, O. E.[1] used the concept that the coefficients of the expected Fisher information matrix as equal to the coefficients of the first fundamental form.

In 1997, Kass, R.E. [2] used the same concept of Barndorff- Nielsen, O.E. using the following formula

$$-\frac{1}{\sqrt{EG}}\left(\frac{\partial}{\partial u}\left(\frac{1}{\sqrt{E}}\frac{\partial\sqrt{G}}{\partial u}\right)+\frac{\partial}{\partial v}\left(\frac{1}{\sqrt{G}}\frac{\partial\sqrt{E}}{\partial v}\right)\right)$$

to compute the Gaussian curvature (K) of trinomial and t families. He gave the general form of a location-scale manifold of density:

$$\left\{p(x)=\frac{1}{v}f\left(\frac{x-u}{v}\right)\middle|(u,v)\in R\times R_{+}\right\}$$

For some density function f.

In 1999, Chen W.W.S. [3] provided a deeper and broader understanding of the meaning of Gaussian curvature, using some more general alternative computational methods. He used the formula

$$\frac{R_{1212}}{EG-F^2}=\frac{(12{,}12)}{EG-F^2}\quad \text{where } (12{,}12)=R_{1212}=\sum_{m=1}^{2}R_{121}^{m}g_{m2}$$

$$R_{ijk}^{1}=\frac{\partial}{\partial u_j}\Gamma_{ik}^{1}-\frac{\partial}{\partial u_i}\Gamma_{jk}^{1}+\Gamma_{ik}^{m}\Gamma_{mj}^{1}-\Gamma_{jk}^{m}\Gamma_{mi}^{1}\text{, sum on } m,$$

to compute the Gaussian curvature (K) for the distributions (normal, Cauchy and t family). He showed that in normal distribution Gaussian curvature $K=-\frac{1}{2}$, and in Cauchy distribution K=-2, while in t family distribution with r degrees of freedom, he get $K=-\frac{r+3}{2r}$. In other words, the Gaussian curvature of t_3 distribution is the geometric mean of the curvatures for the Cauchy and normal distribution.

In 2003, Gruber M.H.J. [4] used the following formula

$$-\frac{1}{2\sqrt{EG-F^2}}\left[\frac{\partial}{\partial u}\frac{G_u-F_v}{\sqrt{EG-F^2}}-\frac{\partial}{\partial v}\frac{F_u-E_v}{\sqrt{EG-F^2}}\right]-\frac{1}{4\left(EG-F^2\right)^2}\begin{vmatrix}E & F & G\\ E_u & F_u & G_u\\ E_v & F_v & G_v\end{vmatrix}$$

to compute the Gaussian curvature of gamma family of distributions and normal distribution. He illustrated some connections between the behaviour of Gaussian curvature of the gamma family of distributions and the central limit theorem as follows:

The random variable that has a Gamma distribution with $\alpha = n$ is the sum of exponential random variables. By the central limit theorem as $n \to \infty$ this random variable tends towards that of a normal distribution. As $n \to \infty$ the curvature of the gamma family of distributions tends towards $-\frac{1}{2}$, the curvature of the normal family of distributions.

In 2004, Arwini K., Del Riego L. and Dodson C. T. J. [5] provided formulae for universal connections and curvatures on exponential families and gave an explicit example for the manifold of gamma distributions.

CHAPTER ONE

Some Definitions and Concepts from Differential Geometry

In this chapter, we introduce some basic ideas and important concepts of differential geometry such as tensor, first fundamental form, second fundamental form, Gaussian curvature, geodesics, etc.

We shall denote the familiar three dimensional Euclidean space (traditionally denoted R^3) as E^3. In studying the geometry of a surface in E^3 we find that some of its most important geometric properties belong to the surface itself and not the surrounding Euclidean space.

The property of a surface which depends only on the metric form is an intrinsic. For example, Gaussian curvature is an intrinsic property of a surface.

1.1 Tensor [6]

An *n*th- rank tensor in *m*- dimensional space is a mathematical object that has *n* indices and m^n components and obeys certain transformation rules. Each index of a tensor ranges over the number of dimensions of space. However, the dimension of the space is largely irrelevant in most tensor equations (with the notable exception of the contracted kronecker delta). Tensors are generalizations of scalars (that have no indices), vectors (that have exactly one index), and matrices (that have exactly two indices) to an arbitrary number of indices.

Tensors provide a natural and concise mathematical framework for formulating and solving problems in areas of physics such as elasticity, fluid mechanics, and general relativity.

The notation for a tensor is similar to that of a matrix (i.e., $A=(a_{ij})$), except that a tensor $a_{ijk\ldots}, a^{ijk\cdots}, a_i^{jk}$, etc., may have an arbitrary number of indices where $i, j, k\ldots=1, 2, \ldots, m$. In addition, a tensor with rank r+s may be of mixed type (r, s), consisting of r so-called “contravariant” (upper) indices and s “covariant” (lower) indices. Note that the positions of the slots in which contravariant and covariant indices are placed are significant so, for example, $a^{\lambda}_{\mu\nu}$ is distinct from $a^{\nu\lambda}_{\mu}$.

While the distinction between covariant and contravariant indices must be made for general tensor, the two are equivalent for tensors in three-dimensional Euclidean space, and such tensors are known as cartesian tensors.

Objects that transform like zeroth-rank tensors are called scalars, those that transform like first- rank tensors are called vectors, and those that transform like second-rank tensors are called matrices. In tensor notation, a vector v would be written v_i, where i= 1, 2, …, m, and matrix is a tensor of type (1, 1), which would be written a_i^j in tensor notation.

Tensors may be operated on by other tensors (such as metric tensors, the permutation tensor, or the kronecker delta) or by tensor operators (such as the covariant or semicolon derivatives). The manipulation of tensor indices to produce identities or to simplify expressions is known as index gymnastics, which includes index lowering and index raising as special cases. These can be achieved through multiplication by a so-called metric tensor g_{ij}, g^{ij}, g_i^j, etc., e.g.,

$$g^{ij}A_j = A^i \qquad i, j = 1, 2, \dots, m \qquad \dots (1.1)$$

$$g_{ij}A^j = A_i \qquad \dots (1.2)$$

The metric tensor is a tensor of rank 2 that is used to measure distance between any two points in a given space.

Tensor notation can provide a very concise way of writing vector and more general identities. For example, in tensor notation, the dot product $u.v$ is simply written

$$u.v = u_i v^i, \qquad \dots (1.3)$$

where repeated indices are summed over (Einstein summation).

Similarly, the cross product can be concisely written as

$$(u \times v)_i = \in_{ijk} u^j v^k, \qquad \text{…. (1.4)}$$

where $\in_{ijk}$ is the permutation tensor

Contravariant second-rank tensors are objects which transform as

$$A'^{ij} = \frac{\partial x'_i}{\partial x_k} \frac{\partial x'_j}{\partial x_l} A^{kl} \qquad i, j, k, l=1, 2, \ldots, m \qquad \text{…. (1.5)}$$

Covariant Second- rank tensors are objects which transform as

$$C'_{ij} = \frac{\partial x_k}{\partial x'_i} \frac{\partial x_l}{\partial x'_j} C_{kl} \quad i, j, k, l = 1,2,\ldots,m \qquad \text{…. (1.6)}$$

Mixed Second- rank tensors are objects which transform as

$$B'^i_j = \frac{\partial x'_i}{\partial x_k} \frac{\partial x_l}{\partial x'_j} B^k_l \quad i, j, k, l = 1,2,\ldots,m \qquad \text{…. (1.7)}$$

where $x'_i = x'_i(x_1, x_2, \ldots, x_m)$ is the coordinate transformation, and $x_i = x_i(x'_1, x'_2, \ldots, x'_m)$ is its inverse.

Definition (1.1.1) [7]

A *second-tensor rank symmetric tensor* is defined as a tensor A for which

$$A^{mn} = A^{nm}$$

Definition (1.1.2) [8]

An *antisymmetric* (also called *alternating*) tensor is a tensor which changes sign when two indices are switched. For example, a tensor $A^{x_1,\ldots,x_n}$ such that

$$A^{x_1 \,,\ldots,\, x_i \,,\ldots,\, x_j,\ldots,\, x_n} = -A^{x_n,\ldots,\, x_i,\ldots,\, x_j,\ldots,\, x_1}$$

where $x_1,\ldots,x_n$ are indices.

The simplest nontrivial antisymmetric tensor is therefore an antisymmetric rank-2 tensor, which satisfies $A^{mn} = -A^{nm}$.

1.2 The First Fundamental Form [9, 10]

Suppose M is a surface determined by $\vec{X}(u,v) \subset E^3$ and suppose $\vec{\alpha}(t)$ is a curve on M, the variable t is called the parameter of the curve, $t \in [a,b]$ for $a,b \in R$. Then we can write $\vec{\alpha}(t) = \vec{X}(u(t),v(t))$ (($u(t)$, $v(t)$)) is a curve in R^2 whose image under $\vec{X}$ is $\vec{\alpha}$). Then

$$\vec{\alpha}'(t) = \frac{\partial \vec{X}}{\partial u}\frac{du}{dt} + \frac{\partial \vec{X}}{\partial v}\frac{dv}{dt} = u'\vec{X}_1 + v'\vec{X}_2 \qquad \text{.... (1.8)}$$

If $s(t)$ represents the arc length along $\vec{\alpha}$ (with $s(a)=0$) then

$$s(t) = \int_a^t \|\vec{\alpha}'(r)\| dr \qquad \text{.... (1.9)}$$

where r is a real variable.

And

$$\frac{ds}{dt} = \|\vec{\alpha}'(t)\| \qquad \text{.... (1.10)}$$

so

$$\left(\frac{ds}{dt}\right)^2 = \|\vec{\alpha}'(t)\|^2 = \vec{\alpha}'.\vec{\alpha}' = \left(u'\vec{X}_1 + v'\vec{X}_2\right).\left(u'\vec{X}_1 + v'\vec{X}_2\right)$$

$$= u'^2\left(\vec{X}_1.\vec{X}_1\right) + 2u'v'\left(\vec{X}_1.\vec{X}_2\right) + v'^2\left(\vec{X}_2.\vec{X}_2\right).$$

Following Gauss' notation (briefly) we denote

$$E = \vec{X}_1.\vec{X}_1, \qquad F = \vec{X}_1.\vec{X}_2, \qquad G = \vec{X}_2.\vec{X}_2 \qquad \text{.... (1.11)}$$

and have

$$\left(\frac{ds}{dt}\right)^2 = E\left(\frac{du}{dt}\right)^2 + 2F\left(\frac{du}{dt}\frac{dv}{dt}\right) + G\left(\frac{dv}{dt}\right)^2 \qquad \text{.... (1.12)}$$

or in differential notation

$$ds^2 = E(du)^2 + 2F(dudv) + G(dv)^2 \qquad \text{.... (1.13)}$$

Definition (1.2.1) [9]

Let M be a surface determined by $\vec{X}(u,v)$. The *first fundamental form* (or more commonly *metric form*) of M is $\left(\frac{ds}{dt}\right)^2$ or $(ds)^2$ as defined in formulas (1.12) and (1.13).

Definition (1.2.2) [10]

The *matrix of the first fundamental form* of a surface M determined by $\vec{X}(u,v)$ is

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix} \equiv \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}$$

where E, F, G are as defined in formula (1.11). This matrix determines dot products of tangent vectors.

If $\vec{v} = a\vec{X}_1 + b\vec{X}_2$ and $\vec{\omega} = c\vec{X}_1 + d\vec{X}_2$ are vectors tangent to a surface M at a given point, then

$$\begin{aligned} \vec{v}.\vec{\omega} &= \left(a\vec{X}_1 + b\vec{X}_2\right).\left(c\vec{X}_1 + d\vec{X}_2\right) \\ &= Eac + F(ad + bc) + Gbd \\ &= \begin{pmatrix} a & b \end{pmatrix} \begin{pmatrix} E & F \\ F & G \end{pmatrix} \begin{pmatrix} c \\ d \end{pmatrix} \end{aligned}$$

Notation [9]

We now replace the parameters u and v with u^1 and u^2 in formula (1.13).

We then have

$$ds^2 = g_{11}\left(du^1\right)^2 + 2g_{12}du^1du^2 + g_{22}\left(du^2\right)^2 = \sum_{i,j} g_{ij}du^idu^j \quad \text{.... (1.14)}$$

where the summation is taken over the set {1, 2}. If $\vec{v}$ is a vector tangent to M at a point $\vec{p}$ and $\vec{v} = \left(v^1, v^2\right)$ in the basis $\left\{\vec{X}_1, \vec{X}_2\right\}$ for the tangent plane at $\vec{p}$, then we have $\vec{v} = \sum_i v^i \vec{X}_i$.

If $\vec{\alpha}(t)$ is a curve on M where $\vec{\alpha}$ is represented by $\vec{X}\left(u^1(t), u^2(t)\right)$ then

$$\vec{\alpha}'(t) = u^{1'}(t)\vec{X}_1 + u^{2'}(t)\vec{X}_2 = \sum_i u^{i'}\vec{X}_i .$$

1.3 The Second Fundamental Form [11]

We have treated a path $\vec{\alpha}(t)$ along a surface M as if it were the trajectory of a particle in E^3. We then interprete $\vec{\alpha}''(t)$ as the acceleration of the particle. Well, a particle can accelerate in two ways:
(1) it can accelerate in the direction of travel, and (2) it can accelerate by changing its direction of travel. We can therefore decompose $\vec{\alpha}''$ into two components, $\vec{\alpha}''_{\vec{T}}$ (representing acceleration in the direction of travel) and $\vec{\alpha}''_{\vec{N}}$ (representing acceleration that changes the direction of travel). We may have dealt with this by taking $\vec{\alpha}''_{\vec{T}}$ as the component of $\vec{\alpha}''$ in the direction of $\vec{\alpha}'$ computed as $\vec{\alpha}''_{\vec{T}} = \left(\vec{\alpha}''.\frac{\vec{\alpha}'}{\|\vec{\alpha}'\|}\right)\frac{\vec{\alpha}'}{\|\vec{\alpha}'\|}$

and $\vec{\alpha}''_{\vec{N}}$ as the "remaining component" of $\vec{\alpha}$ (that is, $\vec{\alpha}''_{\vec{N}} = \vec{\alpha}'' - \vec{\alpha}''_{\vec{T}}$).

The unit tangent vector $\vec{T}(s) = \vec{\alpha}'(s) = u^{i'}\vec{X}_i$ with $\vec{\alpha}$ parameterized in terms of arc length s, $\vec{\alpha} = \vec{\alpha}(s) = \vec{X}\left(u^1(s), u^2(s)\right)$. We can see that $\vec{\alpha}''(s) = \vec{T}'(s)$ is a vector normal to $\vec{\alpha}'$ where $\vec{T}' = k\vec{N}$. We again decompose $\vec{\alpha}''$ into two orthogonal components, but this time we make explicit use of the surface M. We wish to write:

$$\vec{\alpha}'' = \vec{\alpha}''_{\tan} + \vec{\alpha}''_{nor}$$

where $\vec{\alpha}''_{\tan}$ is the component of $\vec{\alpha}''$ tangent to M and $\vec{\alpha}''_{nor}$ is the component of $\vec{\alpha}''$ normal to M. Notice that $\vec{\alpha}''_{\tan}$ will be a linear combination of $\vec{X}_1$ and $\vec{X}_2$ (they are a basis for the tangent plane) and

$\vec{\alpha}''_{nor}$ will be a multiple of the unit normal vector to *M*, $\vec{U}\left(\text{calculated as } \vec{U} = \dfrac{\vec{X}_1 \times \vec{X}_2}{\left\|\vec{X}_1 \times \vec{X}_2\right\|}\right)$.

Since $\vec{\alpha}(s) = \vec{X}\left(u^1(s), u^2(s)\right)$ and $\vec{\alpha}' = u^{i'} \vec{X}_i$ (here, 'means *d*/*ds*), then

$$\vec{\alpha}'' = u^{i''} \vec{X}_i + u^{i'} \vec{X}_i' = u^{i''} \vec{X}_i + u^{i'} \frac{d\vec{X}_i}{ds}$$

Now $u^{i''} \vec{X}_i$ is part of $\vec{\alpha}''_{\tan}$, but $u^{i'} \vec{X}_i'$ may also have a component in the tangent plane. Well,

$$\frac{d\vec{X}_i}{ds} = \frac{d}{ds}\left[\vec{X}_i\left(u^1(s), u^2(s)\right)\right] = \frac{\partial \vec{X}_i}{\partial u^1}\frac{du^1}{ds} + \frac{\partial \vec{X}_i}{\partial u^2}\frac{du^2}{ds}$$

$$= \frac{\partial \vec{X}_i}{\partial u^1} u^{1'} + \frac{\partial \vec{X}_i}{\partial u^2} u^{2'} = \frac{\partial \vec{X}_i}{\partial u^j} u^{j'} .$$

If we denote $\dfrac{\partial^2 \vec{X}}{\partial u^i \partial u^j} = \vec{X}_{ij}$ (we have assumed continuous second partials, so the order of differentiation doesn't matter) then we have $\dfrac{d\vec{X}_i}{ds} = \vec{X}_{ij} u^{j'}$.

So acceleration becomes

$$\vec{\alpha}'' = u^{r''} \vec{X}_r + u^{i'} u^{j'} \vec{X}_{ij} .$$

We now need only to write $\vec{X}_{ij}$ in terms of a component in the tangent plane (and so in terms of $\vec{X}_1$ and $\vec{X}_2$) and a component normal to the tangent plane (which will be a multiple of $\vec{U}$).

Definition (1.3.1) [11]

With the notation above, we define the *formula of Gauss* as

$$\vec{X}_{ij} = \Gamma_{ij}^r \vec{X}_r + L_{ij} \vec{U} . \qquad \text{…. (1.15)}$$

That is we define L_{ij} as the projection of $\vec{X}_{ij}$ in the direction $\vec{U}$. Notice, however, that Γ_{ij}^{r} may not be the projection of $\vec{X}_{ij}$ onto $\vec{X}_r$ since the $\vec{X}_r$'s are not orthonormal.

Note [11]

Since projections are computed from dot products, we immediately have that

$$L_{ij} = \vec{X}_{ij}.\vec{U} = \vec{X}_{ij}.\frac{\vec{X}_1 \times \vec{X}_2}{\left\| \vec{X}_1 \times \vec{X}_2 \right\|} \qquad \text{…. (1.16)}$$

We therefore have

$$\vec{\alpha}'' = \vec{\alpha}''_{\tan} + \vec{\alpha}''_{nor} = (u^{r''} + \Gamma_{ij}^{r} u^{i'} u^{j'})\vec{X}_r + (L_{ij} u^{i'} u^{j'})\vec{U} \qquad \text{…. (1.17)}$$

Definition (1.3.2) [12]

The *second fundamental form* of surface *M* is the matrix $\begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix}$ where the determinate of this matrix is *L*. The projections L_{ij} are defined in formula (1.16).

1.4 Gaussian Curvature [13]

If $f(x, y, z)$ is a (scalar valued) function, then for c a constant, $f(x, y, z)=c$ determines a *surface* (we assume all second partials of *f* are continuous and so the surface is *smooth*). The *gradient* of *f* is $\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right)$.

If $\vec{v}_\circ$ is a vector tangent to the surface $f(x,y,z)=c$ at point $\vec{p}_\circ = (x_\circ, y_\circ, z_\circ)$, then $\nabla f(x_\circ, y_\circ, z_\circ)$ is orthogonal to $\vec{v}_\circ$ (and so ∇f is orthogonal to the surface). The equation of a plane tangent to the surface can be calculated using ∇f as the normal vector for the plane.